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THE OLYMPIC MATHEMATICAL MARATHON

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Chapter I
PROBLEMS

351. Let a, b and c be the lengths of the sides of a triangle with inradius r. Prove that:

$$\frac{a^3}{(a-b-c)^2}+\frac{b^3}{(a-b+c)^2}+\frac{c^3}{(a+b-c)^2}\ge 6\sqrt{3}r.$$

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352. Prove that in any ABC triangle the following relationship is true:

$$a^{2+\sqrt{3}}+b^{2+\sqrt{3}}+c^{2+\sqrt{3}}\ge 3\cdot\left(2r\sqrt{3}\right)^{2+\sqrt{3}}.$$

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353. Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R. Prove that:

$$\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)}\ge\frac{1}{4R^2}.$$

George Apostolopoulos

354. Prove that in any ABC triangle the following relationship is true:

$$a^6+b^6+c^6\ge 81\cdot(2r)^6.$$

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355. Let ABC be a triangle with $\sphericalangle C=2\sphericalangle B$. On side BC consider a point D such that $\sphericalangle BAD=\frac{1}{4}\sphericalangle C$. Prove that:

a. $BD=\dfrac{AB\cdot AC}{AB+AC}$;

b. $\dfrac{\cos C}{\cos B}\ge\dfrac{\cot C}{\cot B}$.

George Apostolopoulos

356. Prove that in any $\Delta\ ABC$ the following relationship is true:

$$\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\le\frac{a^2+b^2+c^2+3}{2\sqrt{2}F}.$$

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357. Let R, r, r_a, r_b, r_c represent the circumradius, inradius and exradii, respectively, of $\Delta\ ABC$. Prove that:

a. $\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)\left(1+\cot\frac{C}{2}\right)\ge\left(1+2\sqrt{3}\frac{r}{R}\right)^3$;

b. $\left(1+\frac{1}{r_a}\right)\left(1+\frac{1}{r_b}\right)\left(1+\frac{1}{r_c}\right)\ge\left(1+\frac{2}{3R}\right)^3$.

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358. Prove that in any $\Delta\,ABC$ the following relationship is true:

$$\sum\left(\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c}\right)^3 \geq \sqrt[3]{3a}+\sqrt[3]{3b}+\sqrt[3]{3c}-2.$$

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359. For an acute triangle ABC and a positive integer n, prove that:

$$\left(\sum(\sin A\sin B\sin C)^{\frac{1}{n}}\right)^n \leq \frac{3^{n+1}}{8},$$

where the sum is over all cyclic permutations of (A,B,C).

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360. Prove that if $m,n \in \mathbb{N}^*$ then in any triangle the following relationship holds:

$$2R\sum w_a^{m+1}(w_a^{2n}-w_a^n+1) \geq \frac{(ab+bc+ca)^{m+n+1}}{(6R)^{m+n}}.$$

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361. Let ABC be a triangle. Prove that:

$$\left(1+\sec\frac{A}{2}\right)^{\csc^{-3}\frac{A}{2}}\cdot\left(1+\sec\frac{B}{2}\right)^{\csc^{-3}\frac{B}{2}}\cdot\left(1+\sec\frac{C}{2}\right)^{\csc^{-3}\frac{C}{2}} \geq \left(5+\frac{26\sqrt{3}}{9}\right)^{\frac{1}{8}}.$$

George Apostolopoulos

362. Prove that in any $\Delta\,ABC$ the following relationship holds:

$$\sum m_a h_a w_a \geq \sin A\sin B\sin C\left(\frac{b^2c^2}{a}+\frac{c^2a^2}{b}+\frac{a^2b^2}{c}\right).$$

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363. Let ABC be a triangle. Prove that:

$$\left(1+\sec\frac{A}{2}\right)^{8\sin^3\frac{A}{2}}\left(1+\sec\frac{B}{2}\right)^{8\sin^3\frac{B}{2}} 2\cdot\left(1+\sec\frac{C}{2}\right)^{8\sin^3\frac{C}{2}} \geq 5+\frac{26}{9}\sqrt{3}.$$

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364. Prove that in any ΔABC the following relationship holds:

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} > \frac{1}{2R}(m_a+m_b+m_c).$$

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365. Let P be an arbitrary point inside a triangle ABC. Let a, b and c be the distances from P to the side BC, AC and AB, respectively. Prove that:

$$\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^4}{\sin^4 A+\sin^4 B+\sin^4 C} \leq 12R^2,$$

where R denotes the circumradius of ABC.

When does the equality occur?

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366. Prove that in ABC triangle the following relationship holds:

$$\sum m_a\left(|\sin B|\sqrt{1+\cos^2 B}+|\cos B|\sqrt{1+\sin^2 B}\right)<2s\sqrt{3}.$$

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367. Let ABC be a given triangle, and let M, N be the interior points on the side BC such that $BM = CN$. Prove that:

$$\frac{1}{AM}+\frac{1}{AN}>\frac{4}{AB+AC}.$$

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368. Prove that in any ABC triangle the following relationship holds:

$$\sqrt{m_a}+\sqrt{m_b}+\sqrt{m_c}\geq\sqrt{m_a+m_b+m_c+6\sqrt[3]{h_a h_b h_c}}.$$

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369. Let ABC be a given triangle, and let M, N be the interior points on the side BC such that $BM = NC$. Prove that:

$$\frac{AC^2}{AM}+\frac{AB^2}{AN}>BC.$$

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370. Prove that in any ΔABC the following relationship holds:

$$\sum\sqrt{a+b-c}\geq\sqrt{2p+12\sqrt[3]{rF}}.$$

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371. In an acute triangle ABC heights AA_1, BB_1, and CC_1 are drawn. Let H be the intersection point of heights. Prove that:

$$\frac{AH}{HA_1}+\frac{BH}{HB_1}+\frac{CH}{HC_1}\geq\sec A+\sec B+\sec C.$$

George Apostolopoulos

372. Prove that in any ABC triangle the following relationship holds:

$$\frac{b^4c^7}{a^{12}}+\frac{c^4a^7}{b^{12}}+\frac{a^4b^7}{c^{12}}\geq\frac{\sqrt{3}}{R}.$$

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373. Let ABC be an acute triangle with inradius r and circumradius R. Prove that:

$$\frac{(\sec A)^{\frac{1}{\sec A}}+(\sec B)^{\frac{1}{\sec B}}+(\sec C)^{\frac{1}{\sec C}}}{\sec A+\sec B+\sec C}<\frac{5R-r}{12r}.$$

George Apostolopoulos

374. Prove that in any ABC triangle the following relationship holds:

$$(m_a^2+m_am_b+m_b^2)(m_b^2+m_bm_c+m_c^2)(m_c^2+m_cm_a+m_a^2)\geq\left(\frac{9r_ar_br_c}{r_a+r_b+r_c}\right)^3.$$

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